University of Basel

Population and economic growth

2. An introduction to OLG models

WWZ

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1. Motivation

In many economic models populations are a homogeneous mass, neglecting important demographic features:

- Individuals age and populations have age structures
- Population movements follow from fertility, mortality and migration

Overlapping generations models are a workhorse framework that enables to study populations in a (more) realistic way.

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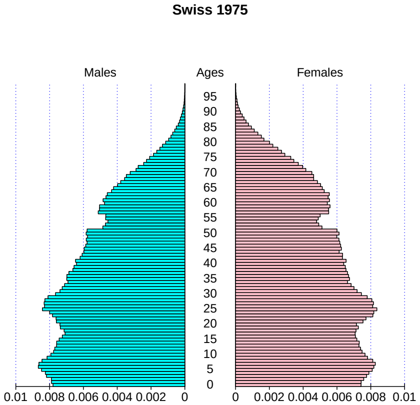

Figure: Swiss Age pyramid
Data source: BFS – STAT-TAB

Why do we care about ageing and age structure?

- Human life-cycle
- Intergenerational transfers
- Social security
- ...

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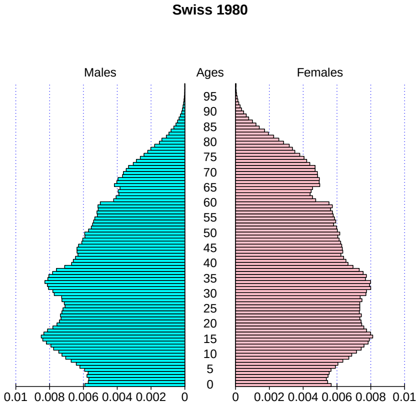

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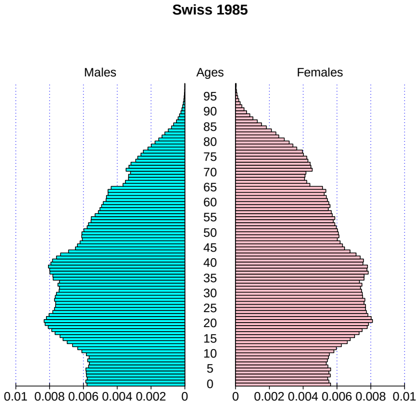

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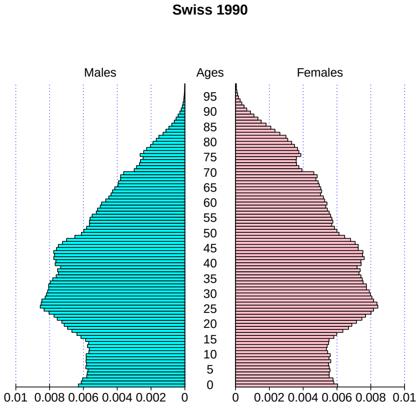

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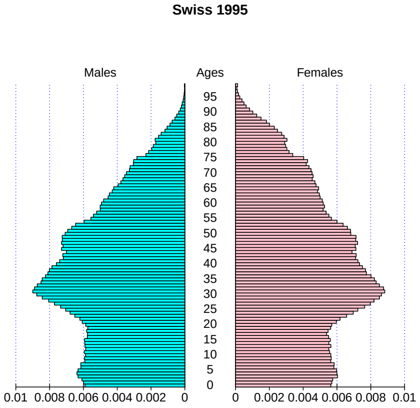

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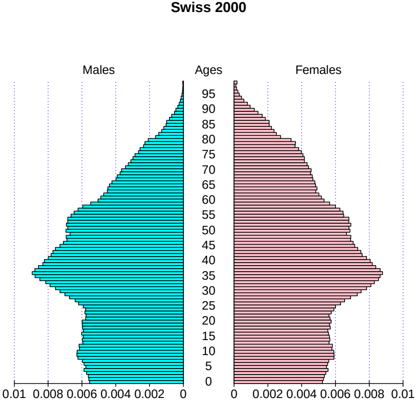

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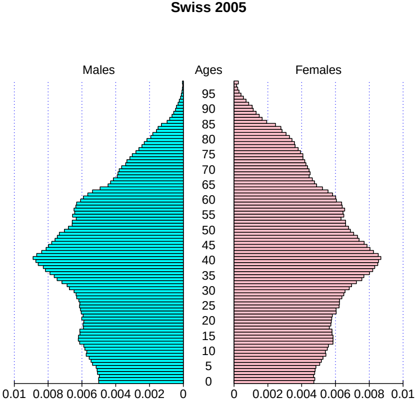

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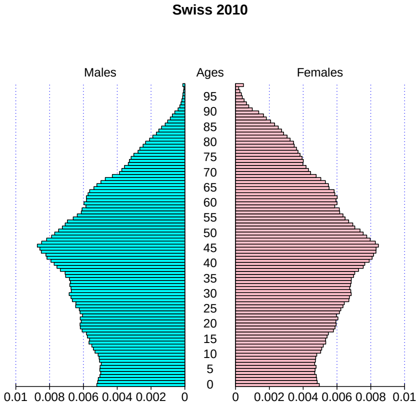

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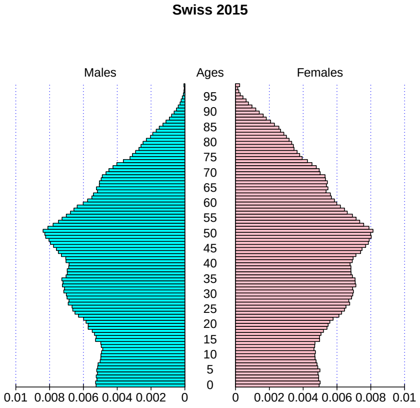

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Outline

2. The general idea

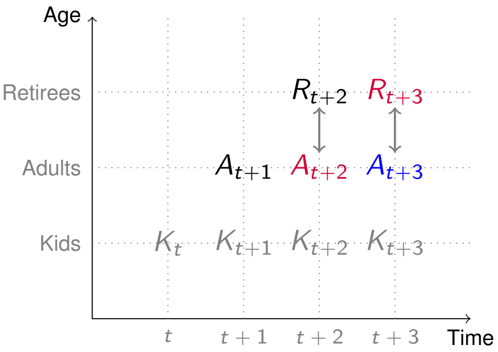
Overlapping generation models (OLG-model) are the second basic micro-based approach in macro economics. OLG-models originate from Allais (1947), Diamond (1965) and Samuelson (1958). They are applied in a wide range of economic fields.

- at each point in time different generations coexist $\rightarrow$ reflects age structures
- Interaction of different generations via markets, transfers...
- We focus on the discrete time model $t = 0, 1, 2, 3, \ldots, \infty$ (The perpetual youth model is a well-known alternative Blanchard (1985)–Yaari (1965))

2. The general idea

Already a 2-period OLG-model enables us to explore many questions:

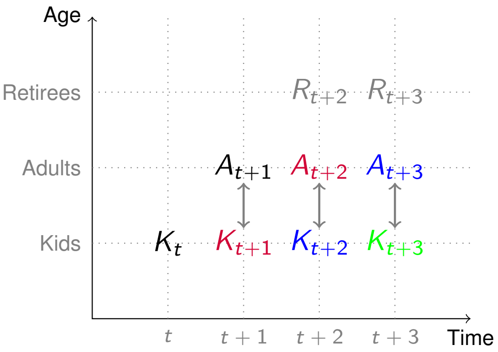
- Working and old individuals
  - Retirement and old-age security
  - Long-term care
  - Longevity
  - ...



2. The general idea

Already a 2-period OLG-model enables us to explore many questions:

- Children and working individuals
  - Fertility decisions
  - Fertility and education
  - Child labour
  - ...



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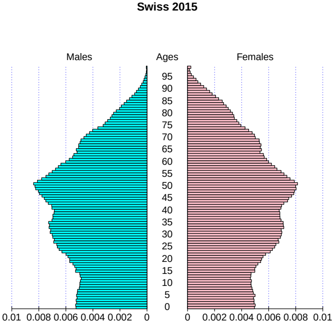

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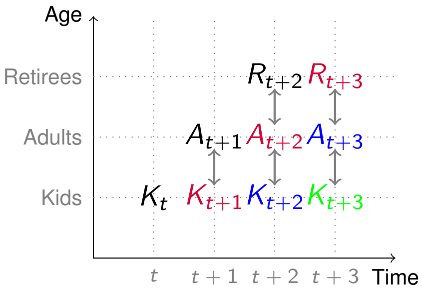


Already 3-period OLG-models with childhood, working-age and retirement reflects the “common” human life-cycle and age structures of populations → Dependency Ratio

2. The general idea

The introduction of finite living interacting individuals has some important implications:

- Opposed to e.g. the Ramsey model, preferences of non-alive generations may not be incorporated in the market transactions.
- Generally, the competitive equilibrium does not coincide with the planner
- The competitive equilibrium may even not be Pareto optimal due to over-accumulation of life-cycle savings.

The basic OLG-model

3. The general framework

Assume a *Robinson Crusoe* island with the following characteristics (our assumptions):

- Closed economy
- No government purchases goods or services
- One sector production technology with a homogeneous final good
- Inputs summarized in Labour L, Capital K (and Technology A)
- Perfect markets with households and firms
- Time is discrete and individuals age and die

3.1 Firms

- Homogeneous firms are active at perfect competitive markets
- Aggregated (neoclassical) production function:

$$Y_t = (K_t, L_t)$$

$$y_t = \frac{Y_t}{L_t} = f(k_t)$$

with $k_t = \frac{K_t}{L_t}$

3.1 Firms

- Profit maximization given the prices for wages w_t and rental prices for capital r_t.

$$\pi = f(k_t) - \delta k_t - r_t k_t - w_t \tag{1}$$

- Competitive markets $\rightarrow$ Labour and capital are compensated by their marginal products:

$$w_t = f(k_t) - k_t f'(k_t)$$

$$r_t + \delta = f'(k_t)$$

It is common to assume $\delta = 1$. The capital stock is fully depreciated every period (of 30 years).

3.2 Population structure

- Individuals live for 2 periods
- Growth rate of cohorts:

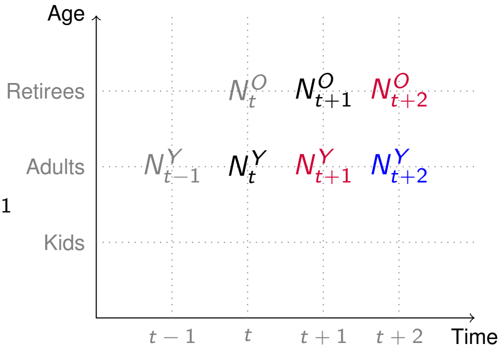
$$N_t^Y = (1+n)^t \, N_0^Y$$

- Population size:

$$N_t = N_t^Y + N_t^O \quad \text{or} \quad N_t = (2+n)\, N_{t-1}^Y$$

- Old-age dependency ratio:

$$DR_t = \frac{N_t^O}{N_t^Y} \quad \text{or} \quad DR_t = \frac{1}{1+n}$$



3.3 Individuals

- Individuals value consumption when young c_t and old d_{t+1}, such that life-cycle utility is captured by

$$U\left(c_t, d_{t+1}\right) = u\left(c_t\right) + \beta u\left(d_{t+1}\right)$$

- β displays the discount factor of future consumption with $\beta = \frac{1}{1+\rho} < 1$ and $\rho > 0$ as notion of pure time preference.
- Concave period utility $u\left(\cdot\right)$: $u'\left(\cdot\right) > 0$, $u''\left(\cdot\right) < 0$

3.3 Individuals

- Individuals are endowed with one effective unit of time supplied on the labour market and compensated by the wage rate w_t
- Young individuals either consume c_t or save s_t their labour income:

$$w_t = c_t + s_t \tag{2}$$

- Old individuals only consume their savings:

$$\left(1 + r_{t+1}^e\right) s_t = d_{t+1} \tag{3}$$

- As only young individuals work, their labour income w_t corresponds to the present value of lifetime spending on consumption: $w_t = c_t + \frac{d_{t+1}}{1+r_{t+1}^e}$

3.3 Individuals

Young individuals in t maximize utility by choosing savings s_t (or consumption in both periods) according to the following maximization problem:

$$\max_{s_t} u\left(c_t\right) + \beta u\left(d_{t+1}\right)$$

subject to eq. (2) and (3) and $s_t, c_t, d_{t+1} \geqslant 0$

First-order condition to this problem is the familiar *Euler Equation*

$$u'\left(c_t\right) = \beta\left(1 + r^e_{t+1}\right) u'\left(d_{t+1}\right)$$

Saving Function

3.4 The Temporary Equilibrium

“Temporary equilibrium is such that all are reaching their ’best’ positions, subject to the constraints by which they are bound, and with the expectations that they have at the moment. Equilibrium over time, if it is to be defined in a corresponding manner, must be such that it is maintainable over a sequence, the expectations on which it is based, in each single period, being consistent with one another.” (Hicks 1965)

3.4 The Temporary Equilibrium

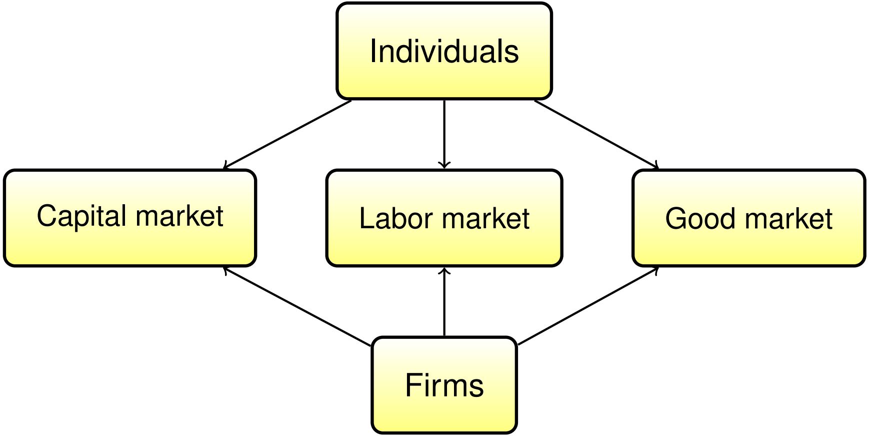


Figure: STRUCTURE OF THE BASIC OLG MODEL

3.4 The Temporary Equilibrium

1. Labour market equilibrium

$$L_t^D = L_t^S = N_t^Y \Leftrightarrow w_t = f(k_t) - k_t f'(k_t) \tag{4}$$

2. Capital market equilibrium

$$I_t = N_t^Y s_t \Leftrightarrow r_t = f'(k_t) - \delta \tag{5}$$

3. Good market equilibrium

$$Y_t = N_{t-1}^Y d_t + N_t^Y (c_t + s_t) \tag{6}$$

3.4 The Temporary Equilibrium

From the capital market equilibrium (eq. (5)) and the law of motion of capital, we know that:

$$I_t = S_t \qquad \text{and} \qquad K_{t+1} = K_t - \delta K_t + I_t$$

As capital is fully depreciated every period $\delta = 1$, it is straightforward to show that $K_{t+1} = S_t$!

3.4 The Temporary Equilibrium

Given the variables from the previous period $\{s_{t-1}, I_{t-1} = N^Y_{t-1} s_{t-1}\}$ and the expected rate of return on savings r^e_{t+1}, the temporary equilibrium of time t is defined by

1. the wage rate w_t and the gross rate of return $1 + r_t$
2. the aggregate variables K_t, L_t, Y_t, k_t and I_t
3. the individual variables $c_t, s_t,$ and d_t

that satisfy the optimality conditions of the agents and the three equilibrium conditions (market equilibria) eq. (4), (5) and (6).

3.5 Dynamics

From the capital accumulation, we can infer:

$$K_{t+1} = N_t^Y s\left(w_t, 1 + r_{t+1}^e\right)$$

Or with perfect foresight $1 + r_{t+1}^e = f'\left(k_{t+1}\right)$ and expressed in terms of per capita:

$$k_{t+1} = \frac{1}{1+n} s\left[w(k_t), 1 + r(k_{t+1})\right]$$

$$k_{t+1} = \frac{1}{1+n} s\left[f\left(k_t\right) - k_t f'(k_t), f'(k_{t+1})\right] \equiv g\left(k_t\right) \tag{7}$$

The *fundamental law of motion of the OLG economy* or *saving locus*

3.5 Dynamics

The properties of the saving locus determine the dynamic behaviour of our economy:

$$\frac{dk_{t+1}}{dk_t} = \frac{\overbrace{-\overbrace{s_w k_t}^{+}\ \overbrace{f''(k_t)}^{-}}^{+}}{\underbrace{1 + n - \underbrace{s_r}_{+/-}\ \underbrace{f''(k_{t+1})}_{-}}_{+/-}}$$

- The sign of $\frac{dk_{t+1}}{dk_t}$ depends on $s_r \lesseqgtr 0$
- If $s_r > 0$, it follows that $\frac{dk_{t+1}}{dk_t} > 0$.

3.5 Dynamics

Inter-temporal equilibrium:

Given an initial capital stock $k_0 = \frac{K_0}{N^Y_{-1}}$, an inter-temporal equilibrium with perfect foresight is a sequence of temporary equilibria that satisfies for all $t \geqslant 0$ the following two conditions:

$$1 + r^e_{t+1} = 1 + r_{t+1} = f'\left(k_{t+1}\right)$$

$$k_{t+1} = \frac{1}{1+n} s\left(w_t, 1 + r_{t+1}\right) \tag{8}$$

3.6 Steady State

A steady state is defined as the stationary solution described by the dynamics of eq. (7), such that $k_t = k_{t+1} = k^*$ and $g\left(k^*\right) = k^*$, or:

$$k^* = \frac{1}{1+n} s\left[f\left(k^*\right) - k^* f'\left(k^*\right), f'\left(k^*\right)\right]$$

The particular case $g\left(0\right) = 0$ is called a corner steady state.

What's about the characteristics of the steady state?

3.6 Steady State

Without further assumptions, we cannot conclude on the existence and uniqueness of Steady States:

- No interior steady state
- A unique interior steady state
- Multiple steady states.

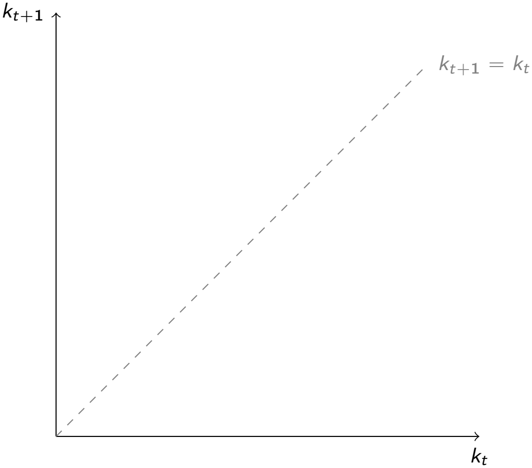
3.6 Steady State

- Local stability requires:

$$\left.\frac{dk_{t+1}}{dk_t}\right|_* = \left|\frac{-s_w k^* f''(k^*)}{1+n-s_r f''(k^*)}\right| < 1$$

- Instability may arise if $s_r < 0$ for instance.
- Oscillatory if $0 > \left.\frac{dk_{t+1}}{dk_t}\right|_* > -1$ and non-oscillatory if $0 < \left.\frac{dk_{t+1}}{dk_t}\right|_* < 1$.

3.6 Steady State



A A corner steady state

B A unique stable interior steady state

C A unique unstable interior steady state

D A stable corner and a stable and unstable interior steady state

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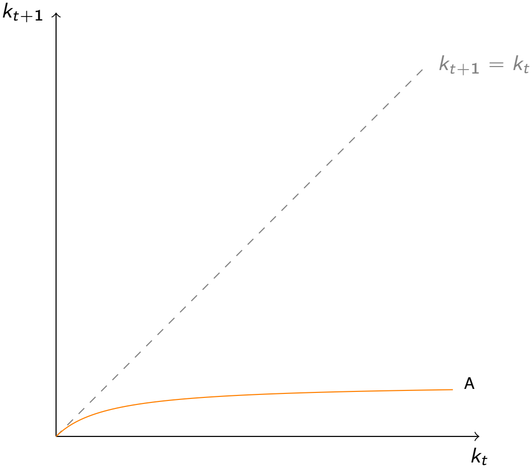

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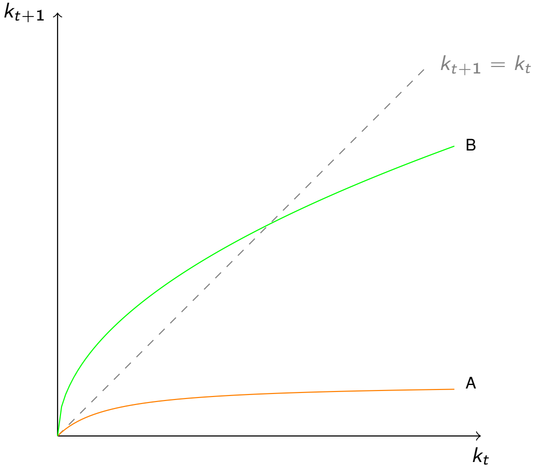

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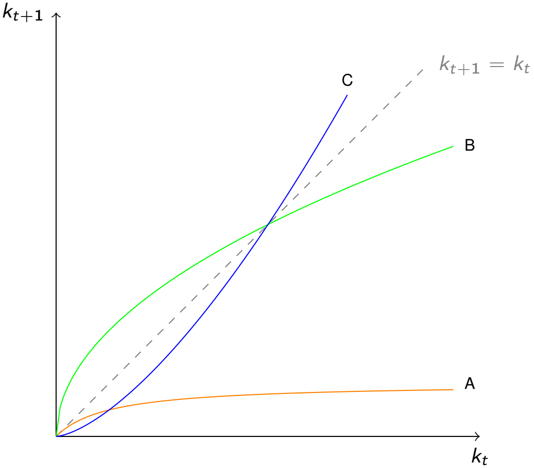

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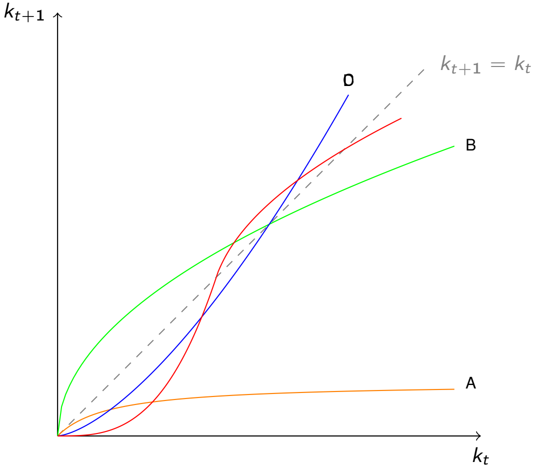

- **A** A corner steady state
- **B** A unique stable interior steady state
- **C** A unique unstable interior steady state
- **D** A stable corner and a stable and unstable interior steady state

4. Specific utility and production function

Let's assume instantaneous utility follows a CRRA function:

$$U_t\left(c_t, d_{t+1}\right) = \frac{c_t^{1-\theta} - 1}{1-\theta} + \beta \frac{d_{t+1}^{1-\theta} - 1}{1-\theta}$$

with $\theta > 0$ and $0 < \beta < 1$.

For simplicity, production follows a Cobb-Douglas technology

$$f\left(k_t\right) = k_t^{\alpha}$$

4. Specific utility and production function

From the profit maximization with $\delta = 1$, we learn:

$$w_t = (1 - \alpha)\, k^{\alpha} \tag{9}$$

and the interest rate:

$$1 + r_t = \alpha k_t^{\alpha - 1} \tag{10}$$

4. Specific utility and production function

The corresponding Euler Equation is:

$$\frac{d_{t+1}}{c_t} = \left(\beta\left(1 + r_{t+1}\right)\right)^{\frac{1}{\theta}}$$

By means of the Budget constraints, we find:

$$\beta\left(1 + r_{t+1}\right)^{1-\theta} s_t^{-\theta} = \left(w_t - s_t\right)^{-\theta}$$

such that

$$s_t = \frac{w_t}{\phi_{t+1}} \tag{11}$$

with $\phi_{t+1} = 1 + \underbrace{\beta^{-\frac{1}{\theta}}\left(1 + r_{t+1}\right)^{-\frac{1-\theta}{\theta}}}_{>0} > 1$

4. Specific utility and production function

- Individuals always save $0 < s_t < w_t$ and savings increase in income:

$$s_w = \frac{1}{\phi_{t+1}}$$

- The curvature of the indifference curve determines if savings increase or decrease with an increasing interest rate:

$$s_r = \left(\frac{1-\theta}{\theta}\right)\left(\beta\left(1+r_{t+1}\right)\right)^{-\frac{1}{\theta}}\frac{s_t}{\phi_{t+1}}$$

  - $\theta > 1 \Rightarrow s_r < 0$: Income effect dominates the substitution effect
  - $\theta = 1 \Rightarrow s_r = 0$: Income and substitution effect balance each other
  - $\theta < 1 \Rightarrow s_r > 0$: Substitution effect dominates income effect

4. Specific utility and production function

To study the dynamics and steady state we plug in eq. (11) into eq. (8):

$$k_{t+1} = \frac{w_t}{(1+n)\,\phi_{t+1}}$$

Taking into account our production technology (eq. (10) and eq. (9)), we obtain:

$$k_{t+1} = \frac{(1-\alpha)\,k_t^{\alpha}}{(1+n)\left[1+\beta^{-\frac{1}{\theta}}\left(\alpha k_{t+1}^{\alpha-1}\right)^{-\frac{1-\theta}{\theta}}\right]} \tag{12}$$

and the steady state for $k_{t+1} = k_t = k^*$

$$(1+n)\left[1+\beta^{-\frac{1}{\theta}}\left(\alpha (k^*)^{\alpha-1}\right)^{-\frac{1-\theta}{\theta}}\right] = (1-\alpha)\,(k^*)^{\alpha-1}$$

4. Specific utility and production function

To investigate stability, we compute the total derivative of eq. (12):

$$\left.\frac{dk_{t+1}}{dk_t}\right|_* = \left| \frac{1}{\frac{1}{\alpha} + (1+n)\left[\frac{1-\theta}{\theta}\left(\beta\alpha(k^*)^{\alpha-1}\right)^{-\frac{1}{\theta}}\right]} \right| < 1$$

Keeping in mind that the saving locus is continuous and monotonic, we can conclude that their exists a unique and stable steady state for any $\theta > 0$.

But: No closed form solutions!

To obtain closed form solutions, many applications focus on the case of log-preferences $\theta = 1$

4. Specific utility and production function

The Canonical OLG model

With logarithmic instantaneous utility $u(\cdot) = \ln \cdot$, the lifetime preferences write:

$$U = \ln(c_t) + \beta \ln(d_{t+1})$$

Maximizing consumption according to the budget constraints (eq. (2) and eq. (3)), yields the *Euler equation*:

$$\frac{d_{t+1}}{c_t} = \beta(1 + r_{t+1}) \tag{13}$$

4. Specific utility and production function

By means of the *Euler equation* (eq. (13)) and the budget constraints (eq. (2) and eq. (3)), we find the following levels on individual consumption:

$$c_t = \frac{1}{1+\beta} w_t$$

$$d_{t+1} = \frac{\beta}{1+\beta} \left(1 + r_{t+1}\right) w_t$$

and savings:

$$s_t = \frac{\beta}{1+\beta} w_t \tag{14}$$

4. Specific utility and production function

We can plug in s_t in the good market equilibrium:

$$k_{t+1} = \frac{1}{1+n} s_t \qquad \Leftrightarrow \qquad k_{t+1} = \frac{\beta w_t}{(1+\beta)(1+n)}$$

$$k_{t+1} = \frac{\beta (1-\alpha) k_t^{\alpha}}{(1+\beta)(1+n)} \tag{15}$$

4. Specific utility and production function

Steady State

Taking into account $k_{t+1} = k_t = k^*$, eq. (15) provides a closed form solution for the steady state:

$$k^* = \left[\frac{\beta (1-\alpha)}{(1+\beta)(1+n)} \right]^{\frac{1}{1-\alpha}}$$

There exists a unique and stable steady state, as:

$$\left. \frac{dk_{t+1}}{dk_t} \right|_* = \alpha < 1$$

Note: As income and substitution effect cancel out in the case of log-preferences, the saving behaviour is *identical* with those of the exogenous savings model.

5. The first-best solution

- To study efficiency (or potential over-accumulation), we explore the first-best solution of a social planner.
- The planner maximizes the weighted average of all (future) generations.

$$\sum_{t=-1}^{\infty} \gamma^t U_t\left(c_t, d_{t+1}\right) \tag{16}$$

  with γ as social discount factor and k_0 and c_{-1} as given.
- Attention: $\gamma < 1$ to ensure concavity and that the objective function is finite.

5. The first-best solution

As the life-cycle utility function is time separable, we can transfer the cohort into the period perspective such that the social welfare function writes as:

$$W = \sum_{t=0}^{\infty} \gamma^t \left(u(c_t) + \frac{\beta}{\gamma} u(d_t) \right)$$

which is maximized w.r.t. the resource constraint:

$$F(K_t, L_t) = I_t + N_t c_t + N_{t-1} d_t$$

or in intensive terms:

$$f(k_t) = c_t + (1+n) k_{t+1} + \frac{1}{1+n} d_t \tag{17}$$

We still assume that capital fully depreciates.

5. The first-best solution

We can solve the optimization problem by plugging in the resource constraint:

$$W = \sum_{t=0}^{\infty} \gamma^t \left[u(c_t) + \frac{\beta}{\gamma} u\left((1+n) \left(f\left(k_t\right) - c_t - (1+n)\, k_{t+1} \right) \right) \right]$$

Differentiating w.r.t c_t and k_{t+1} yields the following two FOC for any interior solution:

$$u'(c_t) = \frac{(1+n)\,\beta}{\gamma} u'(d_t) \tag{18}$$

$$u'(d_t) = \frac{f'(k_{t+1})\gamma}{1+n} u'(d_{t+1}) \tag{19}$$

5. The first-best solution

Combining eq. (18) eq. (19) yields the *Euler equation*.

$$u'(c_t) = \beta f'(k_{t+1})\, u'(d_{t+1})$$

It is the same as in the laissez faire economy! The planner allocates consumption exactly in the same way, as the individuals would do.

Question: Does the planner choose the capital labor ratio in a dynamically efficient way or is “over-accumulation” of capital feasible?

5. The first-best solution

To study this question, let's focus on the "stationary path"

$$U = u(c) + \beta u(d)$$

subject to the resource constraint

$$f(k) = (1+n)k + c + \frac{d}{1+n}$$

furthermore, the non-negative constraints apply.

5. The first-best solution

The highest stationary utility is thus defined by:

$$\max u(c) + \beta u(d) \quad \text{subject to} \quad f(k) = (1+n)k + c + \frac{d}{1+n}$$

Maximizing yields the following set of FOC:

$$f'(k) = 1 + n \tag{20}$$

$$u'(c) = (1+n)\beta u'(d)$$

5. The first-best solution

Eq. (20) is the *Golden Rule of capital accumulation*:

$$f'(k) = 1 + n$$

We can easily illustrate that maximizing life-cycle consumption:

$$\hat{c} = c + \frac{d}{1+n} = f(k) - (1+n)k$$

also yields:

$$\frac{\partial \hat{c}}{\partial k} = f'(k) - (1+n) = 0$$

Without technological progress, the marginal productivity of capital ($f'(k)$) equals the sum of capital depreciation (Remember $\delta = 1$) and population growth n.

5. The first-best solution

Question: Does the social planner reach the Golden Rule level or over-accumulate capital?

From the FOC (eq. (19)) we know that in the steady state:

$$u'(d^*) = \frac{f'(k^*)\gamma}{1+n} u'(d^*) \qquad \Rightarrow \qquad f'(k^*) = \frac{1+n}{\gamma}$$

- If $0 < \gamma < 1$, we can conclude $r^* > n$ and thus $k^* < k^{GR}$
- Only if $\gamma = 1$, the planner realizes the Golden Rule level of capital as she weights the utility of all future generations in the same way.

Note: The First-best solution is pareto-optimal, as the social planner internalizes the effect of savings on future returns to capital.

5. The first-best solution

Question: Are the corresponding *Euler equations* sufficient to ensure that the first-best and laissez-faire economy coincide?

NO! In general savings in the competitive economy do not correspond with the Golden Rule capital labour ratio, or said differently:

$$s \neq (1+n)\, k^{GR}$$

To illustrate this, let's come back to our example with the Cobb-Douglas production technology: $Y_t = AK^{\alpha}L^{1-\alpha}$ and logarithmic instantaneous utility.

5. The first-best solution

Remember, in this case savings, wages and the interest rate follow from eq. (14), (9) and (10), respectively. Then:

$$\frac{\beta}{1+\beta} A\left(1-\alpha\right)\left(k^{GR}\right)^{\alpha} \lesseqgtr \left(1+n\right) k^{GR}$$

taking into account the Golden Rule (eq. (20)), yields:

$$\frac{\beta}{1+\beta} A\left(1-\alpha\right)\left(k^{GR}\right)^{\alpha} \lesseqgtr A\alpha(k^{GR})^{\alpha-1} k^{GR}$$

5. The first-best solution

The laissez-faire generally differs from first-best solution, as:

$$\frac{\beta}{1+\beta} \lesseqgtr \frac{\alpha}{1-\alpha}$$

- With $\alpha, \beta \approx 0.3$ savings are below k^{GR}.
- Only if $\beta = \frac{\alpha}{1-2\alpha}$ the laissez-faire and first-best solution coincide.
- Dynamic inefficiency (over-accumulation) $k^* > k^{\mathrm{GR}}$ might arise from the need of the young to save for old-age consumption.

6. The Serendipity Theorem

Question: How can we ensure that a decentralized economy accumulates capital according to the first-best solution?

Question: What is the optimal growth rate of the population?

Samuelson's **Serendipity Theorem** provides an answer:

It is sufficient to fix population growth to the optimal level to implement the Golden Age. Transfers are not required.

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6. The Serendipity Theorem

Basic Idea for a proof:

Maximization of the optimal stationary path (average utilitarianism)

$$U = u(c) + \beta u(d) \tag{21}$$

subject to the resource constraint

$$f(k) = (1+n)k + c + \frac{d}{1+n} \tag{22}$$

and with respect to the variables k, c, d and $1+n$.

6. The Serendipity Theorem

First-order conditions:

$$f'(k) = 1 + n$$

$$u'(c) = (1+n)\,\beta u'(d) \tag{23}$$

$$d = (1+n)^2\, k \tag{24}$$

Equation (24) describes the optimal choice on population growth $1+n$.

6. The Serendipity Theorem

From the budget constraint of the old, eq. (3) and the market equilibrium eq. (7), we know:

$$d = (1 + r)(1 + n)\,k \quad \text{with} \quad s = (1 + n)\,k$$

If $f'(k) = 1 + r$ and, hence, $1 + r = 1 + n$ holds, the Golden Rule is satisfied, the budget constraint of the old corresponds to eq. (24). Furthermore, eq. (23) becomes:

$$u'(c) = (1 + r)\,\beta u'(d)$$

A transfer is not required to implement the Golden Age.

7. Conclusion

- Overlapping-generations models are a basic microeconomic tool useful to:
  - implement age structure and individual ages
  - to study fertility and mortality
  - transfers,...
- Market solutions do generally differ from the first-best solution.
- Implementing the optimal growth rate of the population implements a Pareto efficient allocation – The Golden Age.

8. Literature

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Questions, comments, suggestions,...?